Lecture Notes, January 27, 2008

General Equilibrium in an Economy with unbounded technology sets

Delete P.VI (bounded $\mathcal{Y}^j$). Like all good mathematicians, we're reducing this to the previous case.

Under assumptions of No Free Lunch (P.IV(a)) and Irreversibility (P.IV(b)), the attainable output set for the economy and for each firm is still bounded.

P.IV. (a) if $y \in Y$ and $y \neq 0$, then $y_k < 0$ for some k.
(b) if .$y \in Y$ and $y \neq 0$, then $-y \notin Y$

Let firm j's (unbounded) production technology be Y^j. Define $S^j(p)$ as j's profit maximizing supply in Y^j. Define $D^i(p)$ as i's demand without restriction to $\{x| |x|\leq c\}$. Note that $S^j(p)$ and $D^i(p)$ may not be well defined.

Define $\tilde{Y}^j = Y^j \cap \{x| |x|\leq c\}$, substitute $\tilde{Y}^j$ for $\mathcal{Y}^j$ in chapters 4 - 7. Define $\tilde{S}^j(p)$ as j's supply function based on $\tilde{Y}^j$.

Theorem 8.3(b): If $\tilde{S}^j(p)$ is attainable, then $S^j(p) = \tilde{S}^j(p)$.

Theorem 9.1(b): If $\tilde{D}^i(p)$ is attainable, then $\tilde{D}^i(p) = D^i(p)$.

$Z(p) = \Sigma_i D^i(p) - \Sigma_j S^j(p) - \Sigma_i r^i$

Theorem 11.1: Assume P.II-P.V, and C.I-C.V, CVII, C.VIII. There is $p^* \in P$ so that p^* is an equilibrium price vector. That is, $Z(p^*) \leq 0$ and $p^*_k = 0$ for k so that $Z_k(p^*) < 0$.

Proof: The artificially bounded economy characterized by production technologies $\tilde{Y}^j$, $j \in F$, is a special case of the bounded economy of chapters 4 - 7. Find equilibrium of that bounded economy. That bounded economy equilibrium is attainable so restriction to length c is not a binding constraint. So bounded and unbounded supply and demand coincide. Equilibrium prices of the bounded economy exist and are equilibrium prices for the unbounded economy with technology sets Y^j. Q.E.D.

Theorem 11.1 here is the most important single result of this course. It says that the competitive economy, guided only by prices, has a market clearing equilibrium outcome. The decentralized price-guided economy has a consistent solution. This is the defining result of the general equilibrium theory.

11.4 The Uzawa Equivalence Theorem

Let S be the unit simplex in R^N . Recall two propositions:

Brouwer Fixed Point Theorem (BFPT): Let $f : S \to S$, f continuous. Then there is $p^* \in S$ so that $p^* = f(p^*)$.

Walrasian Existence of Equilibrium Proposition (WEEP):

Let $X : S \to R^N$ so that

(1) X(p) is continuous for all $p \in S$ and

(2) $p \cdot X(p) = 0$ (Walras' Law) for all $p \in S$.[1]

Then there is $p^* \in S$ so that $X(p^*) \le 0$ with $p_i^* = 0$ for i so that $X_i(p^*) < 0$.

The observation that these two results are equivalent is Theorem 2, below. Mathematical equivalence means that each proposition implies the other. We already know that BFPT implies WEEP; that was Theorem I.2. It remains to demonstrate that the implication goes the other way as well. The proposition requires that ---- using WEEP but not BFPT ---- we prove that for an arbitrary continuous function from the simplex to itself, there is a fixed point. The strategy of proof is to take an arbitrary continuous function f(p) from the simplex into itself. We use f(p) to construct a continuous function mapping from S into R^N fulfilling Walras' Law. That is, we construct an 'excess demand' function (derived from no actual economy but fulfilling the properties required in WEEP). The strategy of proof then is to find the general equilibrium price vector associated with this excess demand function and show that it is also a fixed point for the original function. Obviously this plan requires clever construction of the excess demand function.

[1] We use the strong form of Walras' Law for convenience.

Theorem 2 (Uzawa Equivalence Theorem[2]**)**: WEEP implies BFPT.

Proof: We must demonstrate the following property: Let f(•) be an arbitrary continuous function mapping S into S. Assume WEEP but <u>not</u> BFPT. Then there is $p^* \in S$ so that $f(p^*) = p^*$.

Let $f : S \to S$, f continuous.

Let $\mu(p) \equiv \frac{p \cdot f(p)}{|p|^2}$

$\equiv \frac{|p||f(p)|}{|p|^2} \cos(p, f(p)) \leq \frac{|f(p)|}{|p|}$, where cos(p, f(p)) denotes the cosine of the angle included by p, f(p). Let

$$X(p) \equiv f(p) - \mu(p)p \ .$$

X(p) is the 'excess demand' function.

$p \cdot X(p) = p \cdot f(p) - \frac{p \cdot f(p)}{|p|^2} |p|^2 = 0$; this is Walras' Law (2).

Hence, assuming WEEP, there is $p^* \in S$ so that $X(p^*) \leq 0$. Note that by construction $X(p^*) = 0$. This follows since $p_i^* = 0$ for $X_i(p^*) < 0.$ If there were i so that $X_i(p^*) < 0$, it would lead to a contradiction: $p_i^* = 0$, so $0 > X_i(p^*) = f_i(p^*) - \mu(p^*)p_i^* = f_i(p^*) - 0 \geq 0$.

Therefore $X(p^*) = f(p^*) - \mu(p^*)p^* = 0$.

So $f(p^*) = \mu(p^*)p^*$. But p^* and $f(p^*)$ are both points of the simplex. The only scalar multiple of a point on the simplex that remains on the simplex occurs when the scalar is unity. That is,
$f(p^*) \in S,\ p^* \in S$ and $f(p^*) = \mu(p^*)p^*$ implies $\mu(p^*) = 1,$ which implies $f(p^*) = p^*$.[3]

Q.E.D.

[2] The result is due to Hirofumi Uzawa (1962).
[3] Acknowledgment and thanks to Jin-lung Lin for providing the central idea of this argument.

The Uzawa Equivalence Theorem says that use of the Brouwer Fixed Point Theorem is not merely one way to prove the existence of equilibrium. In a fundamental sense, it is the only way.

Fundamental Theorems of Welfare Economics

12.1 Pareto Efficiency

Definition: An allocation $x^i,\ i \in H$, is attainable if there is $y^j \in Y^j,\ j \in F$ (note change in notation) so that $0 \le \sum_{i\in H} x^i \le \sum_{j\in F} y^j + \sum_{i\in H} r^i$. (The inequalities hold co-ordinatewise.)

Definition: Consider two assignments of bundles to consumers, v^i, $w^i \in X^i$, $i \in H$. v^i is said to be Pareto superior to w^i if for each $i \in H$, $u^i(v^i) \ge u^i(w^i)$ and for some $h \in H$, $u^h(v^h) > u^h(w^h)$.

Note that Pareto preferability is an incomplete ordering. There are many allocation pairs that are Pareto incomparable.

Definition: An attainable assignment of bundles to consumers, $w^i,\ i \in H$, is said to be Pareto efficient (or Pareto optimal) if there is no other attainable assignment v^i so that v^i is Pareto superior to w^i.

Definition: $< p^0, x^{0i}, y^{0j} >,\ p^0 \in R^N_+,\ i \in H,\ j \in F$, $x^{0i} \in R^N$, $y^{0j} \in R^N$, is said to be a competitive equilibrium in a private ownership economy if

(i) $y^{0j} \in Y^j$ and $p^0 \cdot y^{oj} \ge p^0 \cdot y$ for all $y \in Y^j$, for all $j \in F$

(ii) $x^{0i} \in X^i,\ M^i(p^0) = p^0 \cdot r^i + \sum_{j\in F} \alpha^{ij}\, p^0 \cdot y^{0j}$

$p^0 \cdot x^{0i} \le M^i(p^0)$

and $u^i(x^{0i}) \ge u^i(x)$ for all $x \in X^i$ with $p^0 \cdot x \le M^i(p^0)$ for all $i \in H$, and

(iii) $0 \ge \sum_{i\in H} x^{0i} - \sum_{j\in F} y^{0j} - \sum_{i\in H} r^i$

(co-ordinatewise) with $p^0_k = 0$ for co-ordinates k so that the strict inequality holds.

This definition is sufficiently general to include the equilibrium developed in each of Theorems 7.1, 11.1, and 17.7. Properties (i) and (ii) embody decentralization. Property (iii) is market clearing.

12.2 First Fundamental Theorem of Welfare Economics (1FTWE)
Every competitive equilibrium is Pareto efficient (CE ⇒ PE). This result does not require convexity of tastes or technology (though attaining a CE may need convexity).

Theorem 12.1 (First Fundamental Theorem of Welfare Economics): Assume C.II, C.IV. Let $p^0 \in R^N_+$ be a competitive equilibrium price vector of the economy. Let w^{0i}, $i \in H$, be the associated individual consumption bundles, y^{oj}, $j \in F$, be the associated firm supply vectors. Then w^{0i} is Pareto efficient.

Intuition for the proof: Proof by contradiction. If there's a better attainable consumption plan it must be more expensive than CE consumption plan --- evaluated at equilibrium prices. Then it must be more profitable (and attainable) to the firm sector as well. Then it must be available and more profitable to some firm. But that contradicts the definition of CE.

<u>Proof:</u> $u^i(w^{0i}) \geq u^i(x)$, for all x so that $p^0 \bullet x \leq M^i(p^0)$, for all $i \in H$.

• If $u^i(x) > u^i(w^{0i})$, for typical $i \in H$, then $p^0 \cdot x > p^0 \cdot w^{0i}$.

• $p^0 \cdot y > p^0 \cdot y^{oj}$ implies $y \notin Y^j$.

• $\sum_{i \in H} w^{0i} \leq \sum_{j \in F} y^{0j} + r$.

• For each $i \in H$, $p^0 \bullet w^{0i} = M^i(p^0) = p^0 \cdot r^i + \sum_j \alpha^{ij}(p^0 \cdot y^{0j})$, and summing over households,
$$\sum_{i \in H} p^0 \bullet w^{0i} = \sum_i M^i(p^0) = \sum_i \left[p^0 \cdot r^i + \sum_j \alpha^{ij}(p^0 \cdot y^{oj}) \right]$$

$$= p^0 \cdot \sum_i r^i + p^0 \cdot \sum_i \sum_j \alpha^{ij} y^{0j}$$

$$= p^0 \cdot \sum_i r^i + p^0 \cdot \sum_j \sum_i \alpha^{ij} y^{0j}$$

$$= p^0 \cdot r + p^0 \cdot \sum_j y^{0j} \quad \text{(since for each j, } \sum_i \alpha^{ij} = 1 \text{).}$$

Proof by contradiction. Suppose, contrary to the theorem, there is an attainable allocation v^i, $i \in H$, so that $u^i(v^i) \geq u^i(w^{0i})$ all i with $u^h(v^h) > u^h(w^{0h})$ for some $h \in H$. The allocation v^i must be more expensive than w^{0i} for those households made better off and no less expensive for the others. Then we have
$$\sum_{i \in H} p^0 \cdot v^i > \sum_{i \in H} p^0 \cdot w^{0i} = \sum_{i \in H} M^i(p^0) = p^0 \cdot r + p^0 \cdot \sum_{j \in F} y^{0j} .$$
But if v^i is attainable, then there is $y'^j \in Y^j$ for each $j \in F$, so that

$\sum_{i \in H} v^i \leq \sum_{j \in F} y'^j + r$, (co-ordinatewise). But then, evaluating this production plan at the equilibrium prices, p^o, we have

$$p^0 \cdot r + p^0 \cdot \sum_{j \in F} y^{0j} < p^0 \cdot \sum_{i \in H} v^i \leq p^0 \cdot \sum_{j \in F} y'^j + p^0 \cdot r \; .$$

So $p^0 \cdot \sum_{j \in F} y^{0j} < p^0 \cdot \sum_{j \in F} y'^j$. Therefore for some $j \in F,\ p^0 \cdot y^{0j} < p^0 \cdot y'^j$.

But y^{0j} maximizes $p^0 \cdot y$ for all $y \in Y^j$; there cannot be $y'^j \in Y^j$ so that $p \cdot y'^j > p \cdot y^{0j}$. This is a contradiction. Hence, $y'^j \notin Y^j$. The contradiction shows that v^i is not attainable. Q.E.D.

1FTWE does not require convexity.

12.3 Second Fundamental Theorem of Welfare Economics (2FTWE)
(Every PE can be supported as CE subject to income redistribution. Requires convexity). We prove this in two steps, first that there are supporting prices (Thm. 12.2), and second that there is a way to parse endowment and ownership to make budgets balance (this is just bookkeeping, Corollary 12.1).

Recall: **Theorem 2.12 (Separating Hyperplane Theorem)**: Let $A, B \subset R^N$; let A and B be nonempty, convex, and disjoint, that is $A \cap B = \phi$. Then there is $p \in R^N$, $p \neq 0$, so that $p \bullet x \geq p \bullet y$, for all $x \in A$, $y \in B$.

Let $A^i(x^i) \equiv \{x | x \in X^i, \quad u^i(x) \geq u^i(x^i)\}$.

Theorem 12.2: Assume P.I-P.IV and C.I-C.VI. Let x^{*i}, y^{*j} , $i \in H$, $j \in F$, be an attainable Pareto efficient allocation. Then there is $p \in P$ so that

(i) x^{*i} minimizes $p \cdot x$ on $A^i(x^{*i})$, $i \in H$, and
(ii) y^{*j} maximizes $p \cdot y$ on Y^j , $j \in F$.

Proof: Let $x^* = \sum_{i \in H} x^{*i}$, and let $y^* = \sum_{j \in F} y^{*j}$. Note that $x^* \leq y^* + r$ (the inequality applies co-ordinatewise). Let $A = \sum_{i \in H} A^i(x^{*i})$. Let $B = \sum_{j \in F} Y^j + \{r\} = Y + \{r\}$. A and B are closed convex sets with common points, x^*, $y^* + r$.

Let $\mathcal{A} = \sum_{i \in H} \{x \mid x \in X^i, u^i(x) > u^i(x^{*i})\}$. $A = \text{closure}(\mathcal{A})$.

$\mathcal{A}$ and B are disjoint, convex. By the Separating Hyperplane Theorem, there is a normal p , so that $p \bullet x \geq p \bullet v$ for all $x \in \mathcal{A}$, and all $v \in B$. By continuity of u^i, all i, and continuity of the dot product we have also $p \bullet x \geq p \bullet v$ for all $x \in A$ and all $v \in B$ so that $p \bullet x^* \geq p \bullet (y^* + r)$. $p \geq 0$, by (C.IV), and $x^* \leq y^* + r$, so $p \bullet x^* \leq p \bullet (y^* + r)$.

Thus x^* and (y^*+r) minimize $p \bullet w$ on A and maximize $p \bullet w$ on B. Without loss of generality, let $p \in P$. Then --- based on the additive structure of A and B, x^{*i} minimizes $p \bullet x$ on $A^i(x^{*i})$and y^{*j} maximizes $p \bullet y$ on Y^j. That is,

$p \bullet x^* = \min_{x \in A} p \cdot x = \min_{x^i \in A^i(x^{*i})} p \cdot \sum_{i \in H} x^i = \sum_{i \in H} \left(\min_{x \in A^i(x^{*i})} p \cdot x \right)$, and

$p \bullet (r + y^*) = \max_{v \in B} p \bullet v = p \bullet r + \max_{y^j \in Y^j, j \in F} p \cdot \Sigma\, y^j = p \bullet r + \sum_{j \in F} \left(\max_{y^j \in Y^j} p \cdot y^j \right)$. So x^{*i} minimizes $p \bullet x$ for all $x \in A_i(x^{*i})$ and y^{*j} maximizes $p \bullet y$ for all $y \in Y^j$. QED

Corollary 12.1 (Second Fundamental Theorem of Welfare Economics): Assume P.I-P.IV, and C.I-C.VI. Let x^{*i}, y^{*j} be an attainable Pareto efficient allocation. Then there is $p \in P$ and a choice $\hat{r}^i \geq 0$, $\hat{\alpha}^{ij} \geq 0$ so that

$$\sum_{i \in H} \hat{r}^i = r$$

$$\sum_{i \in H} \hat{\alpha}^{ij} = 1 \text{ for each j, and}$$

$$p \cdot y^{*j} \text{ maximizes } p \cdot y \text{ for } y \in Y^j$$

$$p \cdot x^{*i} = p \cdot \hat{r}^i + \sum_{j \in F} \alpha^{ij}(p \cdot y^{*j})$$

and (Case 1, $p \cdot x^{*i} > \min_{x \in X^i} p \cdot x$) $u^i(x^{*i}) \geq u^i(x)$ for all $x \in X^i$ so that $p \cdot x \leq p \cdot \hat{r}^i + \sum_{j \in F} \hat{\alpha}^{ij}(p \cdot y^{*j})$

or (Case 2, $p \cdot x^{*i} = \min_{x \in X^i} p \cdot x$) x^{*i} minimizes $p \cdot x$ for all x so that $u^i(x) \geq u^i(x^{*i})$.

Proof: By Theorem 12.2, there is $p \in P$ so that y^{*j} maximizes $p \cdot y$ for all $y \in Y^j$, and so that x^{*i} minimizes $p \cdot x$ for all $x \in A^i(x^{*i})$.

By attainability,

$\sum_{i \in H} x^{*i} \leq \sum_{j \in F} y^{*j} + r$. Multiplying through by p, with the recognition of free goods, we have

$$\sum_{i \in H} p \cdot x^{*i} = \sum_{j \in F} p \cdot y^{*j} + p \cdot r$$

Let $\lambda_i = \frac{p \cdot x^{*i}}{\sum_{h \in H} p \cdot x^{*h}}$, and set $\hat{r}^i = \lambda_i r$, $\hat{\alpha}^{ij} = \lambda_i$, for all $i \in H$, $j \in F$. Then

$$p \cdot x^{*i} = p \cdot \hat{r}^i + \sum_{j \in F} \hat{\alpha}^{ij} p \cdot y^{*j} \quad .$$

Now show that cost minimization subject to utility constraint is equivalent to utility maximization subject to a budget constraint (in case 1). This follows from continuity of u^i. Suppose, on the contrary, there is x'^i so that $p \cdot x'^i = p \cdot x^{*i}$ and $u^i(x'^i) > u^i(x^{*i})$. By continuity of u^i, C.V, there is an ε neighborhood about x'^i so that all points in the neighborhood have higher utility than x^{*i} . But then some points of the neighborhood are less expensive at p than x^{*i} , and x^{*i} is no longer a cost minimizer for $A_i(x^{*i})$. This is a contradiction, hence there can be no such x'^i.

The assertion for case 2 is merely a restatement of the property shown in Theorem 2.

QED